ULTRASONIC CAUSTICS AND THE INVERSE SCATTERING PROBLEM IN NDE

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ABSTRACT

This paper explores theoretically the possibility of using caustics, formed in the ultrasonic field diffracted by defects, as an approach to the inverse scattering problem. The case of crack-like defects is considered in detail, using the geometrical theory of diffraction.

The involute of the far field caustic reproduces the projection of the crack edge in the incident beam direction, for a plane incident wavefront. This purely geometrical inversion is carried out uniquely for the astroid and its involute, the elliptical edge. For a general edge shape, the complete inversion requires one further length measurement, which may be carried out in some cases by further experiments with caustics. Useful limitations on the possible shapes of caustics are explained on the basis of catastrophe theory. Calculations show that the inherent intensity-level change (~2-3 d B) and width (~ wavelength) over which it occurs for a typical ultrasonic caustic are adequate for observation. Some discussion is given of experimental requirements, as well as of caustics formed in the near field of a crack and of those formed by voids and inclusions. The topology of the far field caustic cannot in general distinguish between volumetric and crack-like defects. Studying caustics may prove to be a useful adjunct to ultrasonic imaging systems for the inspection of fatigue cracks.

INTRODUCTION

Recently, attention has been given to the theoretical inversion of ultrasonic scattering data (1-4), so that the characteristics of the scattering singularity can be identified. This paper will explore the possibility of using caustics, which are the envelopes of rays diffracted by the defect, for this inversion. Caustics lie in or near the geometrical shadow of the defect; they also occur, in principle, in the back-scattered field, though this region is not studied here because of the anticipated lower contrast, and the likely experimental complications.

THEORY

The scattering of high frequency ultrasound by defects can be described by the geometrical theory of diffraction (5). This theory gives the asymptotic wave amplitude u diffracted to a field point $\underset{\sim}{y}$ from a point $\underset{\sim}{x}$ on the edge of a crack as a series in decreasing powers of the wave number k. The first and most important term is

$$u(\underset{\sim}{y}) \sim A\left[\frac{r}{(r+\sigma)\sigma}\right] \exp\left[i k \phi(\underset{\sim}{\sigma}, s)\right] \quad (1)$$

where $\underset{\sim}{\sigma} = \underset{\sim}{y} - \underset{\sim}{x}$, s measures arc length along the edge, A is am amplitude factor, and r is the distance from the edge to the caustic along the ray (Fig. 1). If $\phi_0(s)$ is the phase of the incident wave at s,

$$\phi(\underset{\sim}{\sigma}, s) = \phi_0(s) + \sigma \quad (2)$$

If more than one geometrical ray path passes from edge points through $\underset{\sim}{y}$. (1) should be replaced by a sum. When $\underset{\sim}{y}$ lies near the caustic, $(r+\sigma) \to 0$ and (1) should be replaced by a superposition of plane waves (6)

$$u(\underset{\sim}{y}) = \int_{edge} \gamma(\underset{\sim}{\sigma}, s) \exp\left[i k \phi(\underset{\sim}{\sigma}, s)\right] ds \quad (3)$$

For large k, (3) can be evaluated by the stationary phase method, which states that the dominant contributions to $u(\underset{\sim}{y})$ come from points on the edge where the derivative $\phi_s = 0$. If β is the angle between the incident ray and the tangent to the edge at s (Fig. 1) the field at $\underset{\sim}{y}$ is due to rays which satisfy the condition

$$\phi_s(\underset{\sim}{\sigma}, s) = \cos\beta - \underset{\sim}{\sigma} \cdot \underset{\sim}{t}/\sigma = 0 \quad (4)$$

The envelope of these cones of rays, which is the caustic surface of the singly diffracted rays, satisfies (4) and also the equation

$$\phi_{ss} = \frac{\partial(\cos\beta)}{\partial s} - \frac{\underset{\sim}{\sigma} \cdot \underset{\sim}{n}}{\sigma \rho} + \frac{\sin^2\beta}{\sigma} = 0 \quad (5)$$

where $\underset{\sim}{n}$ is the principal normal and ρ is the radius of curvature of the edge at s. Eqn. (5) corresponds to the coalescence of two stationary phase points along the edge. In the far field, Eqns. (4) and (5) (which describe the caustic surface) reduce to those defining the caustic surface produced by the projection of the object in the incident beam direction (5). Using i to denote this projection, the far field caustic is then defined by

$$\underset{\sim}{\sigma}^i \cdot \underset{\sim}{t}^i = 0 \quad (6)$$

$$\underset{\sim}{\sigma}^i \cdot \underset{\sim}{n}^i = \rho^i \quad (7)$$

Eqn. (7) shows that the far field caustic contains the evolute of the projection of the edge, i.e. the locus of the centres of curvature of points on this projection. Eqn. (6) shows that the far field caustic surface is a cylinder with generators in the incident beam direction, so that in the classical limit $k \to \infty$ for which these interpretations strictly hold, all far field cross-sections are identical. Therefore, if the geometry of the far field caustic in ultrasonics can be observed, the shape and, in some cases, the size (as discussed below) of the crack projection in the incident beam direction can be found simply by constructing the involute of the caustic. Because (6) and (7)are independent of k , the geometries of caustics formed by S and P waves are identical, so that the present scalar wave analysis is adequate for waves in solids as well as in liquids.

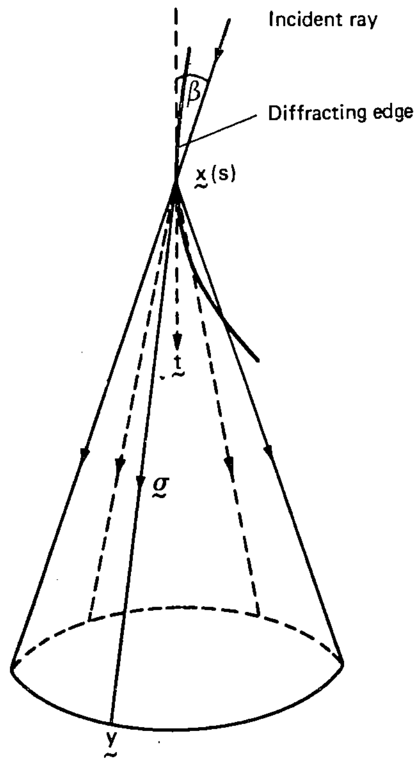


Fig. 1 The cone of rays diffracted from the incident ray at the point $\underset{\sim}{x}(s)$ on the edge, and the geometrical definition of parameters used.

Any theoretical limitations that can be placed on the possible shapes of caustics would greatly assist their identification. For this purpose, we adapt the application of the catastrophe theory of THOM (7) to wave phenomena (8). Because the local description of the crack edge requires only one parameter (s), only cuspoid catastrophes are generically ('typically') possible for edge diffraction. Since the far field depends on two co-ordinates (control parameters), only elementary folds, which appear as ordinary points of the caustic, and cusps, can occur. It follows that the far field caustic produced by diffraction from a purely convex edge projection consists of a closed line interrupted only by cusps. Departures from this rule can occur by accident or by symmetry, but are unlikely for diffraction by real cracks. In the near field, the next cuspoid catastrophe (the so-called 'swallow-tail') is generically possible, though likely to be difficult to observe in ultrasonics because of the compressed form of its near-singular sections.

Cusps correspond to the coalescence of three geometrical ray paths, or three stationary phase points on the edge projection, which occurs when $\phi_{sss} = 0$. Using (7) with the Fresnet formulae, it follows that $\rho_s' = 0$ for cusps in the far field caustic, i.e., cusps occur when the curvature or radius of curvature of the corresponding points on the projected edge is extremal. Also, the cusp is normal to the edge at this corresponding extremal point.

RECONSTRUCTION OF THE DIFFRACTING OBJECT

Consider firstly a diffracting edge whose projection is an ellipse with principal axes 2a and 2b. Then, the cross-section of the far field caustic will be the evolute shown in Fig. 2, which is an astroid. Conversely, when the caustic is observed to be an astroid, it is immediately known that the edge projection is elliptical. If the distances between the two pairs of opposite cusps are measured as $2\xi_0$ and $2\eta_0$, it follows from the equation of the astroid that the major axis of the ellipse is $2a = 2\xi_0\eta_0^2/(\eta_0^2 - \xi_0^2)$, and $b/a = \xi_0/\eta_0$. Since the cusps are normal to the tangents at the corresponding extremal points on the edge, the ellipse is oriented as shown in Fig. 2 with its major axis parallel to the line $2\xi_0$ between the closer pair of cusps. Thus it is actually not necessary to observe the complete caustic in this simple case-only the positions of the cusps in the rectangular array are required.

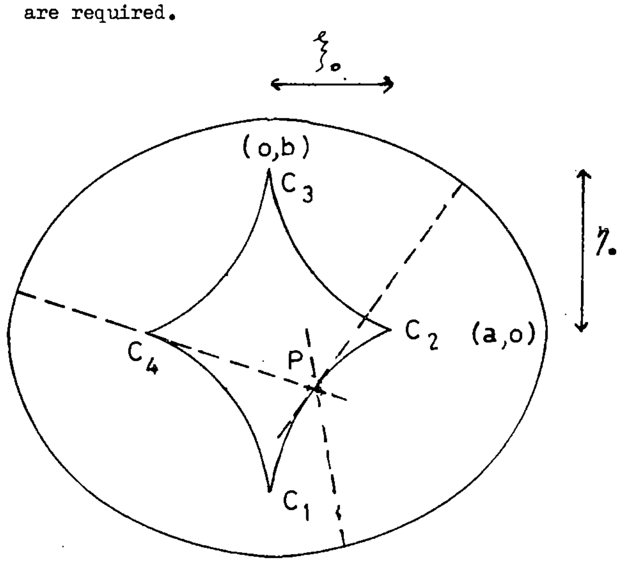
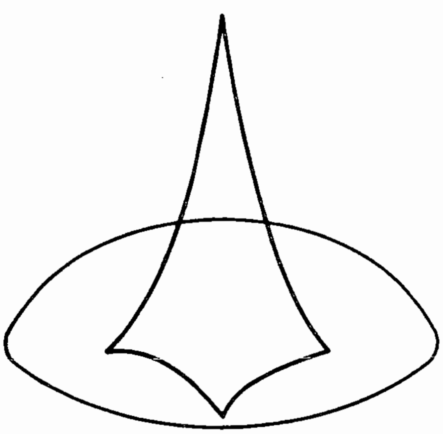


Fig. 2 An elliptical diffracting edge and the corresponding far field caustic. This figure represents a superposition of the spaces of the edge projection and of the far field diffraction pattern, drawn on the same scale. The two cusps lying along the major axis are always inside the geometrical shadow; the other two cusps are outside the shadow for ellipses having eccentricity $> 1/\sqrt{2}$. The dashed lines indicate all rays contributing to the field at point P, as discussed below.

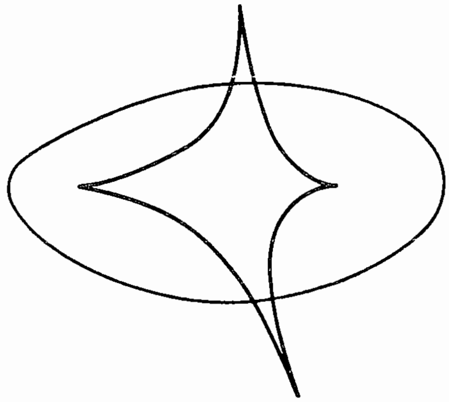
It is useful to construct the involute of the astroid in another way, based on the knowledge that the caustic is the locus of centres of curvature of the edge. Imagine a string set along the inside of the section C_1C_2 of the astroid (Fig. 2) and extending beyond C_2 in the direction of the cusp by a distance equal to the minimum radius of curvature of the ellipse, which is known from the caustic to be $b^2/a = \xi_0^3/(\eta_0^2 - \xi_0^2)$. Unwinding this string traces out the first quadrant of the ellipse. Next, wind the string onto the section C_1C_4 of the astroid to produce the second quadrant of the ellipse. Proceeding clockwise around the astroid and alternately winding and unwinding in this way, the complete ellipse is generated in an anticlockwise sense.

Some special cases of the ellipse are of interest. For eccentricity $\epsilon \to 1$, $\xi_0 \to a$ so that $\eta_0 \to \infty$, i.e. one pair of cusps extends laterally to infinity and is not observed. Therefore, tilting the object about the axis of the closer pair of cusps enables direct measurement of the major axis, since the two remaining cusps become coincident with the ends of the narrow shadow boundary. Also, for $\epsilon \to 0$, the edge approaches a circle and $\xi_0 \simeq \eta_0 \to 0$, giving the diffracted spot at the centre of the shadow, which is well known in optics to have an intensity equal to that of the incident beam. Observation of such a degenerate caustic immediately gives the projection of the diffracting edge as being circular. This particular case is not described in the classification given by Thom's theorem, because of its high symmetry. The size of a nearly circular defect may be found by tilting the specimen, giving an elliptical projection whose caustic is an astroid of convenient dimensions.

Some other examples of caustic/diffracting edge pairs can be generated from the elliptical case. If one side of a distorted ellipse is flatter than the other, a caustic will result which has two pairs of cusps arrayed at right angles, but not symmetrically (Fig. 3a). Again, if an ellipse is distorted by shifting one turning point from the symmetrical position, the cusps will no longer be directed as two opposing pairs at right angles (Fig. 3b). Nevertheless, the directions of the tangents at the turning points are immediately known by inspection of the caustic.



3(a)



3(b)

Fig. 3 Distorted ellipses with their corresponding caustics.

(a) One side of the ellipse more eccentric than the other.

(b) One minimum of curvature shifted from the symmetrical position.

If the edge projection contains a concave part, the caustic will appear discontinuous, since it extends to infinity at points of inflection. Figure 4 shows an ellipse 'pressed in' at one end, and the corresponding caustic. Note that the intensity of the caustic tends to zero far out along those sections which are asymptotic to the normal at the points of inflection, because the density of contributing ray paths then approaches zero. Therefore, only a limited part of the caustic will actually be observed. A less severe depression in the end of the ellipse will produce the cusp marked D further away from the rest of the caustic. A 'flattened end' on an ellipse which is nevertheless convex will give a caustic section as a closed line containing six cusps. Three of these cusps coalesce into one as the distortion of the ellipse is reduced to zero, again producing the simple 4- cusped astroid.

Involuting the far field caustic gives the projection of the diffracting edge in the incident beam direction. The orientation of a planar defect in three dimensions could be inferred from several such measurements involving different projections. Another approach is to examine the variation of the caustic pattern as the plane of observation is moved into the near field: the special case of normal incidence produces no change of the caustic in this region. Identifying this behaviour defines the normal to a planar defect.

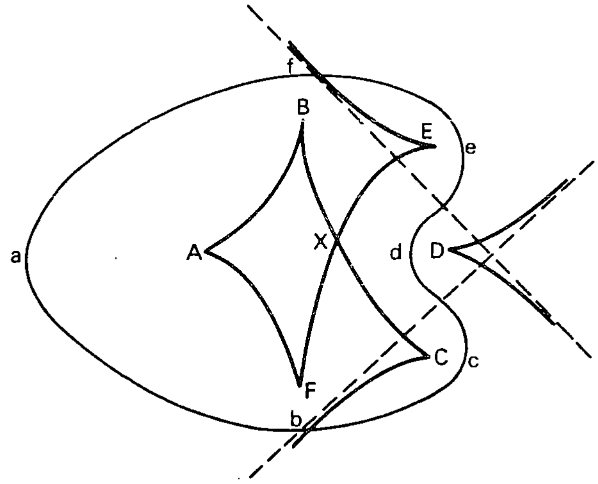


Fig. 4 An ellipse 'pressed in' at one end, and its corresponding caustic. The association between cusps and turning points is indicated by letters. The pattern in the region marked X is a superposition of elementary fold catastrophes, and should not be mistaken for a section of some higher catastrophe.

Any smooth, purely convex closed shape has an even number of turning points, since maxima and minima of curvature must alternate in circuit. Therefore, there is an even number of cusps in the caustic which is itself a partial check on an experiment. The diffracting edge projection is normal to the cusps at the points of extremal curvature, so the orientation of the projection is known by inspection of the cusps. Edge projections corresponding to caustics of four, six, eight or higher even number of cusps can be constructed by alternately folding and unfolding each caustic section in turn, just as was done for the ellipse. For this general case, there is no representation of the caustic in terms of elementary functions as there was for the astroid. Therefore, the radius of curvature at one extremal point cannot be deduced simply from the spacing of cusps, and a different method must be sought to achieve a unique reconstruction.

Since the cusps corresponding to minima of $\rho(s)$ are less sharp than those corresponding to maxima (e.g. see the ellipse of Fig. 2), it should be possible to identify at least one cusp corresponding to a minimum of $\rho(s)$, say ρ_1. Beginning with this cusp, and assuming particular values for ρ_1, a one-parameter family of possible involutes of the caustic can be generated. It then remains to choose the correct involute from this set. One technique suitable for planar defects would be to tilt the object about an axis between two approximately opposite cusps, whose spacing would then be asymptotic to the length of the narrow shadow boundary. This procedure enables direct measurement of one length in the diffracting edge, which is sufficient to select the correct involute. If one length in the edge can be determined by a different technique, as is possible for some objects using ultrasonic spectroscopy, the desired involute can again be chosen from the set of possibilities.

CALCULATED CONTRAST

The ease of observation of caustics in ultrasonics will depend on their width as well as their intensity relative to the background in their neighbourhood. These properties will be studied by considering a particular but typical case, viz. normal incidence of a plane wavefront on a planar elliptical crack. Calculations will be made for a point on the caustic which lies within the geometrical shadow of the ellipse; the contrast will be low for any part of the caustic which lies outside this shadow in the bright field of the incident wave.

Referring again to Fig. 2, the coalescence of two stationary phase points at $\theta = \pi/4$, for example, contributes to the caustic surface along a line which projects into the astroid segment C_1C_2 at point P. In addition, these rays from $\theta = \pi/4$ contribute as from an isolated stationary point to caustic segments in the first and third quadrants, respectively before and after passing through the caustic. Equivalently, there are two 'isolated' contributions from the fourth and second quadrants to the field at all points along the caustic line through P. The question then reduces to a comparison between the intensity on the 'bright side' of the caustic (the inside of the astroid), to which four rays contribute including two which coalesce, and the intensity on the 'dark side' to which two rays contribute.

Table 1 lists results of calculations which make use of the stationary phase evaluation of the diffraction integral for the isolated contributing rays at point P, and the transitional approximation for the coalescing rays. These results are for the caustic 100 mm behind the ellipse (a, b) = (10,7.5) mm at 10MHz for water and for typical P and S wavelengths in steel. C_{max} is the maximum contrast, defined as the ratio of the intensity at the first peak of the Airy fringes which 'clothe' the caustic on the bright side, to the intensity on the dark side. C_{av} represents the contrast predicted by averaging over the first two fringes. The spacings of these fringes in ultrasonics are of the order of the wavelength λ, which is also an estimate of the resolution that can be achieved in any scanning or imaging system which may be used to observe caustics. Therefore the Airy fringes will not readily be observed in ultrasonics, particularly when broadband transducers are used, so C_{av} gives a more realistic estimate of the expected caustic contrast than C_{max}. The changes of intensity level ~ 2-3dB in table 1 are observable. The widths d_w over which these changes occur are ~ λ or slightly greater, which is also suitable for observation.

	Water	Steel (S)	Steel (P)
(mm)	0.15	0.30	0.60
C_{max}	3.12	2.68	2.32
(dB)	(4.94)	(4.28)	(3.65)
C_{av}	2.02	1.79	1.64
(dB)	(3.04)	(2.56)	(2.14)
d_w (mm)	0.24	0.37	0.60

Table 1 Intensity changes across the caustic at point P expressed as a ratio and in dB, and caustic width, listed at 10MHz for the section 100 mm behind an elliptical crack having semi-major axes (a,b) = (10,7.5) mm.

DISCUSSION

Thus far, this paper has assumed a plane incident wavefront. If this wavefront is curved, as for example from a point source, the interpretation of the caustic is more complicated because its geometry depends on the curvature of the incident wave as well as on that of the edge. Nevertheless, the caustic is just as sharp and easily detected. The most important factor experimentally is to tailor the incident wave to minimize the angular spread of wavelets incident on a single point on the edge, since this spread smears out the caustic.

One approach to forming the incident field would be to use a focused ultrasonic probe, with its minimum spot set at the back focal plane of an acoustical lens to produce the convenient (though not essential) plane wavefront. Alternatively, the minimum spot could be produced by a normal probe together with another acoustical lens, which in practice can reduce the width of the generated sound field to the order of λ (9). Other approaches may be to exploit either the direct production of a plane wavefront from a piezoelectric plate (10), or the low divergence of beams of Gaussian cross-section (11). Ultrasonic point sources of diameter 10 μm or less have been generated using lasers (12) and, at the expense of complexity, these sources appear most promising for observing ultrasonic caustics. An initial demonstration of ultrasonic caustics may be most easily carried out in the near field, since the lateral smearing would increase with distance from the defect. For the actual observation of the diffracted field, any scanning or imaging system with sufficient resolution could be used.

The caustic pattern is found in and near the geometrical shadow, and its dimensions are typically comparable to those of the defect. Therefore, the study of these patterns is not seen as a means of improving the resolution of imaging systems. The advantage may come in dealing with defects which, though sufficiently large, produce images that cannot be easily interpreted. For example, an image of a fatigue crack can be complicated by specular reflection from facets on the crack faces and by penetration through regions of crack closure. However, if the edge itself is opaque to ultrasound, as may be inferred from studies of the crack tip in Al alloys (13), the caustic will still be formed. 'False' edges formed at regions of closure are much smaller than the true edge, and would give caustics which were smaller and either unresolved or else easily distinguished from the caustic from the true edge.

In the near field of rays transmitted by a solid or liquid filled inclusion, it can be shown that the caustic sections described as elliptic or hyperbolic umbilics are expected, though these are likely to be masked by diffraction effects and experimental smearing. In the far field, again only elementary folds and cusps are generically possible. Therefore, the topology of far field caustics cannot distinguish between inclusions and planar defects. This result does not hold if the orientation of the specimen is regarded as an additional accessible control parameter in the sense discussed in (8); then, singular umbilic sections can in principle be produced in the far field by rotating the specimen.

A situation similar to the above occurs for voids, for which umbilic sections in the near field result from the decay of creeping waves, but these are not likely to be observable. The far field again consists only of elementary folds and cusps. An important singular case is the spherical void, which gives a point caustic at the centre of its shadow for all orientations. This case is not described by THOM's theorem because of its high symmetry.

A more detailed account of this work will be published elsewhere.

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SUMMARY DISCUSSION
(P. A. Doyle)

Brian DeFacio (Session Chairman-Ames Laboratory): We have time for a couple of questions. Let me ask one. I thought I remember there was a dilation or change of scale between the star in the middle and the ellipsoid. Is that wrong?

Peter Doyle: You mean the ellipse, the figure?

Brian DeFacio: And the caustic imposed. Is it unique, or is there a dilatation involved?

Peter Doyle: You're worrying that the caustic is smaller ..

Brian DeFacio: I'm worrying that outside some minimal ellipse I can put some family of ellipse around --

Peter Doyle: We end up with a family of ellipses, or more general shape. We end up with a family starting from the small size right up to the big size, and we have to make it one length. Does that answer your question?

Brian DeFacio: Yes.

J. D. Auchenbach (Northwestern University): I'm not an experimentalist, but I'm a little bit pesimistic because it seems that you would have to scan the whole field to detect the position of these caustics, and particularly since experimentalists work in the time delay, as a wave starts to move back and forth the position of caustics also there is an evolution in time. If you have a crack, of course, the edge diffraction will produce one in three dimensions, the surface in space. If it isn't normal. Then if surface waves start to propagate over the surface of the crack, they produce another system of diffracted waves which in turn produces its caustic. Of course the intensity decreases as time goes on. All this takes place as time evolves. Since you have to be everywhere in space because you have to map out a number of points to get back to this caustic, it all seems to be a difficult problem.

Peter Doyle: It takes too long to date out, but this is the caustic of the singly diffracted wave. And that will turn up in the geometrical shadows. There is no way to look for that. The point is, caustic is not really scattered waves, which is maybe a surface wave in the meantime. They turn up in different directions.

J. Auchenbach: That may well be so, but, see, you don't know in advance where to look.

Peter Doyle: I have a beam coming down here, and whatever orientation is, it's over in the shadow. You don't have to know that. The orientation. It's the projection of the edge that you learn about in the farfield projection of the edge and the incident beam direction.

J. Auchenbach: You don't know in advance where the crack is, and the crack is very small. You know the incident beam, but you don't know the shadows on it.

Peter Doyle: I should have said we assumed we know where the crack is. We're trying to find out how big it is, its shape and size.

K. K. Galveston (SMU): Unless you work at extremely high frequencies, the bright spots on the disk are within the boundary layer of the shadow boundaries, and so the hole will get smeared out unless you're using high frequencies like in the optics range, where they have the class of experiments where you can actually see them. In the elastomagnetic case, you're working at three, four, five -- what is it? Wave number three, four five or something like that. And the boundary is computed from the edge and will smear out the caustics involved. Especially if you have bright spots inside the crack.

Peter Doyle: One particular case of the degenerative caustic is the case of a circular crack.

K. Galveston: But even then the same thing happens, the bright spot gets smeared out of the boundary layers.

Peter Doyle: It's not an easy experiment, but this is a proposal.

(continued)

P. Doyle (continued discussion)

K. Galveston: You have to have a frequency range within (inaudible).

Peter Doyle: These populations are only ten megahertz.

K. Galveston: They are only valid if you include the effect of the boundary layers, the shadows. You have to include the effects of the shadow boundary. That boundary layer also has to be included in the calculation because they coalesce.

Peter Doyle: Most of the caustic won't be at the shadow boundary. The most important feature is the cusp.

J. Auchenbach: I think if you know where the crack is -- I discussed this a couple of years ago with Wolfgang Sachse -- I don't know if he is around here. If you know where the crack is, I think -- I don't know. You were going to try it.

Wolfgang Sachse (Cornell University): I was going to tell that, but I guess I wanted to let them finish. I had a couple undergraduates about three years ago try this experiment of having a circular disk approximately six inches. We also tried four inches, eight inches in diameter. And we were moving a 40 kilohertz transducer. This was in a large room. And they were measuring the amplitude as a function of distance behind the disk. They were mapping out an XY grid behind the disk. And the only configuration of obstacle in which we were able to see some really significant intensity, if you will, variation behind it was the disk. And in that case -- I don't have the slide with me, but in that case I remember that we had a very, very bright spot behind the disk. We also did ellipses, various shapes. In fact, we used the same technique that this 1908 paper used to make the ellipse. But we got some results. But you really had to use your imagination to say there was a bright region in a particular -- behind the disk, behind the ellipse.

Peter Doyle: Could I make one comment to that -- The K.A. value in that case worked out -- as a matter of fact, ten short of some experiment (inaudible).

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